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On stability of perturbed impulsive differential systems

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Abstract

The notion of Lipschitz stability of impulsive systems of differential equations (DEs) was introduced. In this paper, we will extend the notion of eventual stability to impulsive systems of DEs and extend the notion of Lipschitz stability of impulsive systems of DEs to a new type of stability called eventual Lipschitz stability. Some criteria and results are given. Our technique depends on Liapunov's direct method and comparison principle.
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1. Introduction

The qualitative properties in the mathematical theory of impulsive systems of differential equations have been very important, of interest and developed by a large number of mathematicians, see [1–3,6], and their studies have attracted much attention. Furthermore they have been successful in different approaches based on Liapunov's direct method and comparison technique (see [5]). In recent years the study of such systems has been very intensive (see the monographs [1–3,6], and their bibliographies).

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Lipschitz stability notion is extended to impulsive systems of differential equations by Kulev and Bainov [2,3]. The notion of eventual stability was introduced for ordinary differential equations (see [4]).

Our purpose of this paper is to extend the notion of eventual stability to impulsive systems of differential equations and extend the notion of Lipschitz stability of [2] to a new type of stability of impulsive systems, namely eventual Lipschitz stability. These notions will be studied for the perturbed systems with bounded perturbed function. These notions lie somewhere between Lipschitz stability of [2] on one side and eventual stability on the other side. Furthermore the notion of eventual Lipschitz stability implies both the notions of eventual stability and Lipschitz stability.

Let R^n-be the n-dimensional Euclidean real space, and

$$S(\rho) = \{x \in R^n \colon \|x\| < \rho, \rho > 0\}.$$

The following definitions will be needed.

Definition 1.1 (*Vasundhara*, 1993 [6]). Let $0 \leqslant t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, and $t_k \to \infty$ as $k \to \infty$. Then we say that $F \in PC[R^+ \times R^n, R^m]$ if $F : [t_{k-1}, t_k] \times R^n \to R^m$ is a *continuous function* in $[t_{k-1}, t_k] \times R^n$ and for every $x \in R^n$

$$\mathrm{Lim}_{(t,y)\to(t_k^+,x)} F(t,y) = F(t_k^+, x)$$

exists for $k = 1, 2, \ldots$

Definition 1.2 (*Vasundhara*, 1993 [6]). We say that $V \in V_0$ if $V \in PC[R^+ \times S(\rho), R^+]$, $V(t,x)$ is locally Lipschitzian in x for $(t,x) \in [t_{k-1}, t_k] \times S(\rho)$.

Definition 1.3 (*Vasundhara*, 1993 [6]). A function $\phi(r)$ is said to belong to the class $\aleph$ if $\phi(r) \in C[(0,\rho), R^+]$, $\phi(0) = 0$ and $\phi(r)$ is strictly monotone increasing in r.

Consider the impulsive systems

$$\begin{aligned} &y' = f(t,y),\\ &\Delta y\,|_{t=t_k} = I_k(y),\\ &y(t_0+0) = x_0 \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &x' = f(t,x) + h(t,x),\\ &\Delta x\,|_{t=t_k} = I_k(x) + J_k(x),\\ &x(t_0+0) = x_0, \end{aligned} \tag{1.2}$$

where $f, h \in PC[R^+ \times R^n, R^n]$, $I_k, J_k : R^n \to R^n$, $0 \leqslant t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, and $t_k \to \infty$ as $k \to \infty$.

Going through [6], we define a Liapunov function $V \in PC[R^+ \times S(\rho), R^+]$, and for any fixed $t > t_0$, the function

$$D_{-}V(s,y)(t,s,x) = \mathrm{Lim}_{\delta\to 0} + \inf\left(\frac{1}{\delta}\right)[V(s+h, y(t, s+h, x+\delta h(s,x))) - V(s, y(t,s,x))], \tag{1.3}$$

for $t_0 < s \leqslant t,\ s \neq t_k$, $x \in S(\rho)$, and as in [4], we define the function

$$D^{+}V(t,x) = \mathrm{Lim}_{\delta\to 0} + \sup\frac{1}{\delta}[V(t+\delta, x+\delta f) - V(t,x)].$$

The following assumption will be needed:

(H_1) The solutions $y(t,t_0,x_0)$ and $x(t,t_0,x_0)$ of (1.1) and (1.2) exist for all $t \geqslant t_0 \geqslant \tau$, unique, continuous with the same initial values and $\|y(t,t_0,x_0)\|$, $\|x(t,t_0,x_0)\|$ are locally Lipschitzian in x_0.

(H_2) $h(t,x)$ is a bounded function.

Now, we state the results of [5,6] without their proofs. The following comparison shows the results of [6], which is an important tool for relating the solutions of (1.2) to the solution (1.1).

Theorem 1.1. *Let the hypothesis* (H_1) *be satisfied. Suppose further that* $V \in V_0$ *and*

(i) *For* $t_0 < s \leqslant t$, $s \neq t_k$, *and* $x \in S(\rho)$,

$$D_{-}V(s,y)(t,s,x) \leqslant g(s, V(s, y(t,s,x))). \tag{1.4}$$

(ii) *There exists a* $\rho_0 = \rho_0(\rho) > 0$ *such that* $\|x\| < \rho_0 \Rightarrow \|x - I_k(x)\| < \rho$, *and*

$$V\big(t_k^+, y\big(t, t_k^+, x(t_k) + I_k(x(t_k))\big)\big) \leqslant \psi_k(V(t_k, y(t, t_k, x(t_k)))), \tag{1.5}$$

where $\psi_k : R^+ \to R^+$ *are nondecreasing functions for all* k.

(iii) $g \in PC[R^+ \times R^+, R]$ *and the maximal solution* $r(t) = r(t,t_0,x_0)$ *of the scalar impulsive differential equation*

$$\begin{aligned} &u' = g(t,u), \quad t \neq t_k, \\ &u(t_k+0) = G_k(u(t_k)), \\ &u(t_0+0) = u_0 \geqslant 0 \end{aligned} \tag{1.6}$$

exists for $t \geqslant t_0$. *Then if* $x(t) = x(t,t_0,x_0)$ *is any solution of* (1.2) *and* $V(t_0^+, y(t, t_0^+, x_0)) \leqslant u_0$, *we get*

$$V(t, x(t,t_0,x_0)) \leqslant r(t,t_0,x_0), \quad t \geqslant t_0. \tag{1.7}$$

Lemma 1.1 (Lakshmikantham, 1989 [5]). *Consider*

$$\begin{aligned} &u' = p(t)\phi(u), \quad t \neq t_k, \\ &u'(t_k^*) = G_k(u(t_k)), \\ &u(t_0) = u_0 \geqslant 0, \end{aligned} \tag{1.8}$$

where $p \in C[R^+, R^+]$, ϕ, $G_k \in \kappa$, *and suppose that there exists* $p_0 > 0$ *such that for every* $\sigma \in (0, \rho_0]$,

$$\int_{t_k}^{t_{k-1}} p(s)\,\mathrm{d}s \leqslant \int_{\sigma}^{G_k(\sigma)} \frac{\mathrm{d}s}{\phi(s)} \leqslant 0, \quad k = 1, 2, \ldots \tag{1.9}$$

Then the zero solution of (1.8) *is stable.*

The following definitions will be needed in the sequel.

Definition 1.4 (*Bainov and Simeonov*, 1989 [1]). The zero solution of (1.1) is said to be *uniformly stable* if for every $\epsilon > 0$, $t_0 \in R^n$, $t_0 \geqslant 0$ such that

$$\|x_0\| < \delta \;\Rightarrow\; \|x(t, t_0, x_0)\| < \epsilon, \quad t \geqslant t_0.$$

Definition 1.5 (*Kulev and Bainov*, 1993 [2]). The zero solution of (1.1) is said to be *asymptotically in variation* if for $t \geqslant t_0 \geqslant 0$, there exists a $D > 0$ such that

$$\int_{t_0}^{t} \|\psi(t, s)\|\,\mathrm{d}s \leqslant D$$

and

$$\sum_{t_0 \leqslant t_k < t} \|\psi(t, t_k + 0)\| \leqslant D.$$

The following definitions are somewhat new and related with those of [2,4].

Definition 1.6. The zero solution of the system (1.1) is said to be *uniformly eventually Lipschitz stable* if for $\epsilon > 0$, there exist $M > 0$, $\delta(\epsilon) > 0$, and $\tau(\epsilon) > 0$ such that $\|x_0\| \leqslant \delta$, $x_0 \in R^n$, implies $\|x(t, t_0, x_0)\| \leqslant M\|x_0\|$, $t \geqslant t_0 \geqslant \tau(\epsilon)$. Any eventual Lipschitz stability notions can be similarly defined.

Definition 1.7. The zero solution of the system (1.1) is said to be *uniformly eventually stable* if for $\epsilon > 0$, there exist $M > 0$, $\delta(\epsilon) > 0$, and $\tau(\epsilon) > 0$ such that

$$\|x_0\| \leqslant \delta \;\Rightarrow\; \|x(t, t_0, x_0)\| \leqslant \epsilon, \;\; t \geqslant t_0 \geqslant \tau(\epsilon), \;\; x_0 \in R^n.$$

Definition 1.8. The zero solution of (1.1) is said to be *uniformly eventually asymptotically stable* if it is uniformly eventually stable, and for $\epsilon > 0$, there exist $\delta(\epsilon) > 0$, $\tau(\epsilon)$, and $T(\epsilon) > 0$ such that for $x_0 \in R^n$

$$\|x_0\| \leqslant \delta \;\Rightarrow\; \|x(t, t_0, x_0)\| \leqslant \epsilon, \; t \geqslant t_0 + T(\epsilon) \text{ and } t_0 \geqslant \tau(\epsilon).$$

Any eventually stability notions can be similarly defined.

Remark 1.1. For Definitions 1.6 and 1.7, if the zero solution of (1.1) is uniformly eventually Lipschitz stable, then it is uniformly Lipschitz stable and is uniformly eventually stable.

Consider the impulsive variational systems of (1.1)

$$\begin{aligned} &y' = f_x(t, 0), \quad t \neq t_k, \\ &\Delta y \,|_{t=t_k} = I_k'(0)y, \\ &y(t_0 + 0) = y_0 \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} &z' = f_x(t, x(t, t_0, x_0))z, \quad t \neq t_k \\ &\Delta z \,|_{t=t_k} = I_k'(x(t_k, t_0, x_0))z \\ &z(t_0 + 0) = z_0. \end{aligned} \tag{1.11}$$

Furthermore, we consider the linear impulsive system

$$\begin{aligned} &x' = A(t)x, \quad t \neq t_k, \\ &\Delta x \,|_{t=t_k} = B_k x, \\ &x(t_0 + 0) = x_0, \end{aligned} \tag{1.12}$$

where $f_x = (\partial f / \partial x)$, $I_k'(x) = (\partial I_k / \partial x)$ and $x(t, t_0, x_0)$ be any solution of (1.1) satisfying the initial condition $x(t, t_0, x_0) = x_0$, and A is an $n \times n$ matrix defined in J, and B_k, $k = 1, 2, \ldots$, are constant $n \times n$ matrices.

The fundamental matrix solution $\Phi(t, t_0, x_0)$ of the system (1.11) is defined by

$$\Phi(t, t_0, x_0) = \frac{\partial x(t, t_0, x_0)}{\partial x_0}, \quad t \neq t_k \tag{1.13}$$

(see [5, Theorem 2.4.1]).

The fundamental matrix solution $w(t,s)$ of the system (1.12) is defined by

$$w(t,s) = \begin{cases} u(t,s), \\ \quad t_{k-1} < s < t < t_k, \\ u(t,t_k)(E+B_k)u(t_k,s), \\ \quad t_{k-1} < s < t_k < t \leqslant t_{k+1}, \\ u(s,t_{k+1})\prod_{j=i}^{1}(E+B_{k+j})u(t_{k+j},t_{k+j-1})(E+B_k)u(t_k,s), \\ \quad t_{k-1} < s \leqslant t_k < t_k < t_{k+1} < t \leqslant t_{k+i+1}, \end{cases} \tag{1.14}$$

where E is the unit $n \times n$ matrix, and $u(t,s)$ is the fundamental matrix solution of the system (1.12) without impulses.

2. Main results

In this section, we discuss the notions of eventual stability and Lipschitz stability of impulsive systems of differential equations (1.1) and (1.2).

Theorem 2.1. *Let the hypothesis of Theorem* 1.1 *and the assumption* (H_1) *be satisfied. Assume further that*
(iv) $f(t,0) = h(t,0) = g(t,0) = 0$, *and* $I_k(0) = J_k(0) = \psi_k(0)$ *for all* k.
(v) $b\|x\| \leqslant V(t,x) \leqslant a\|x\|$, $a, b \in \aleph$ *for* $(t,x) \in R^+ \times S(\rho)$.
If the zero solution of (1.1) *is uniformly eventually stable*, *and the zero solution of* (1.6) *is uniformly eventually asymptotically stable*, *then the zero solution of* (1.2) *is uniformly eventually asymptotically stable.*

Proof. Let $y(t,t_0,x_0)$ be a solution of (1.1) with the initial values x_0. From the hypothesis (H_1) and since the zero solution of (1.1) is uniformly eventually stable for $\epsilon > 0$, given $\delta_2(\epsilon) > 0$, there exists $\delta_0 = \delta_0(\epsilon) > 0$ such that

$$\|x_0\| \leqslant \delta_0 \quad \Rightarrow \quad y(t,t_0,x_0)\| \leqslant \delta_2 \tag{2.1}$$

for $t \geqslant t_0 \geqslant \ \geqslant \tau(\epsilon)$. Let $0 < \epsilon\rho^* = \min(\rho_0, \rho)$ be given, $t_0 \in R^+$. Since the zero solution of (1.6) is uniformly eventually stable, given $b(\epsilon) > 0$, $t_0 \in R$, there exist $\delta_1 = \delta_1(\epsilon) > 0$, and $\tau(\epsilon) > 0$ such that

$$0 < u_0 < \delta \quad \Rightarrow \quad u(t,t_0,u_0) < b(\epsilon). \tag{2.2}$$

Let the solutions $y(t,t_0,x_0)$ and $x(t,t_0,x_0)$ of (1.1) and (1.2) with the same initial values x_0, respectively, by using the variation of constant formula, the solutions of (1.1) and (1.2) with the same initial values are related by

$$x(t,t_0,x_0) = y(t,t_0,x_0) + \int_{t_0}^{t} \phi(t,s,x(s,t_0,x_0))\,\mathrm{d}s,$$

where $\phi(t,t_0,x_0)$ is the fundamental matrix solution of (1.11)

$$\|x(t,t_0,x_0)\| = \|y(t,t_0,x_0)\| + \int_{t_0}^{t} \|\phi(t,s,x(s,t_0,x_0))h(s,x(s,t_0,x_0))\|\,\mathrm{d}s. \tag{2.3}$$

From (2.1), inequality (2.3) becomes

$$\|x(t,t_0,x_0)\| = \epsilon + \int_{t_0}^{t} \|\phi(t,s,x(s,t_0,x_0))h(s,x(s,t_0,x_0))\|\,\mathrm{d}s.$$

By using Bellman's inequality, we get

$$\|x(t,t_0,x_0)\| = \epsilon + \int_{t_0}^{t} \gamma(s)\,\mathrm{d}s < \epsilon$$

whenever $\|x_0\| < \delta_0$, $t \geqslant t_0 \geqslant \tau(\epsilon)$. Therefore the zero solution of (1.2) is uniformly eventually stable.

Now, to prove that the zero solution of (1.2) is uniformly eventually asymptotically stable, it is required to prove that $\|x_0\| < \delta_0$ which implies

$$\|x(t,t_0,x_0)\| < \epsilon, \quad t \geqslant t_0 + T, \quad t \geqslant t_0 \geqslant \tau(\epsilon), \quad T > 0, \tag{2.4}$$

where $x(t,t_0,x_0)$ is any solution of (1.2).

If this is not true, then there exists $t^* > t_0 > \tau$, such that $\|x_0\| < \delta_0$ implies

$$\|x(t^*)\| \geqslant \epsilon \quad \text{and} \quad \|x(t)\| < \epsilon \quad \text{for } t_0 < \tau < t < t_k.$$

Then $\|x(t_k)\| < \epsilon\rho_0$ and hence by condition (ii) we get

$$\|x(t_k^+)\| = \|x(t_k) + I_k(t_k)\| < \rho.$$

Hence, we can find t^0 such that $t_k < t^0 \geqslant t^*$ satisfying

$$\epsilon \leqslant x(t^0) < \rho, \tag{2.5}$$

thus for $t_0 \leqslant t \leqslant t^0$, $t_0 > \tau(\epsilon)$, $x(t) < \rho$, and therefore by Theorem 1.1, we get

$$V(t,x(t,t_0,x_0)) \leqslant r(t,t_0,y(t,t_0,x_0)). \tag{2.6}$$

Choose $\delta_2 = a^{-1}(\delta_1)$, thus by using (2.1), (2.2), (2.5), (2.6), and condition (v), we obtain

$$\begin{aligned} b(\epsilon) &\leqslant b\|x(t^0,t_0,x_0)\| \leqslant V(t^0,x(t^0,t_0,x_0)) \leqslant r(t^0,t_0,V(t^0,y(t^0,t_0,x_0))) \\ &\leqslant r(t^0,t_0,a\|y(t^0,t_0,x_0)\|) \leqslant r(t^0,t_0,a(\delta_2)) \leqslant r(t^0,t_0,\delta_1) \leqslant b(\epsilon). \end{aligned}$$

This is a contradiction and then the zero solution of (2.2) is uniformly eventually stable.

Now, let the zero solution of (1.6) be uniformly eventually asymptotically stable. Therefore given $b(\epsilon) > 0$, $t_0 \in R^+$ and $t_0 \geqslant \tau(\epsilon)$, there exist $\delta_1 > 0$, and $T(\epsilon) > 0$ such that $u_0 < \delta_1$ implies

$$u(t,t_0,u_0) \leqslant b(\epsilon), \quad t \geqslant t_0 + T, \quad t_0 \geqslant \tau(\epsilon). \tag{2.7}$$

From (2.3), by choosing $\epsilon = \rho/2$, it follows that: Choose $\delta = \min[\delta_1, \delta_0]$, and let $\|x_0\| < \delta$.

From our previous arguments and condition (v) it yields

$$\begin{aligned} b\|(x(t,t_0,x_0))\| &\leqslant V(t,x(t,t_0,x_0)) \leqslant r(t,t_0,V(t,y(t,t_0,x_0))) \leqslant r(t,t_0,a(\delta_2)) \\ &\leqslant r(t,t_0,\delta_1) \leqslant b(\epsilon) \end{aligned}$$

for $t \geqslant t_0 + T$, $t_0 \geqslant \tau(\epsilon)$. Hence the zero solution of (1.2) is uniformly eventually asymptotically stable, and the proof is completed. □

Theorem 2.2. *Let there exist a function* $V \in C[R^+ \times S(\rho), R^+]$, *and* $V(t,0) = 0$, *such that*

(vi) $D^+V(t,x) \leqslant g(t,V(t,x))$,

(vii) $\|V(t,x) - V(t,y)\| \leqslant L\|x-y\|$, $L > 1$.

(viii) $b\|x\| \leqslant V(t,x)$, $b^{-1}(\alpha x) \leqslant q(\alpha)x$,

where $q(\alpha) \geqslant 1$, $\alpha \geqslant 1$, *for some function* $q, b \in \kappa$.

If the zero solution of (1.6) *is uniformly eventually Lipschitz stable*, *then so is the zero solution of* (1.1).

Proof. Let the zero solution of (1.6) be uniformly eventually Lipschitz stable. Then for every $\epsilon > 0$, $t_0 > 0$, $0 < \epsilon < \rho$, $\rho > 0$, there exist $M > 1$, $\delta_1 > 0$, and $\tau(\epsilon) > 0$ such that

$$0 < u_0 < \delta_1(\epsilon) \quad \Rightarrow \quad u(t,t_0,u_0) < Mu_0, \;\; t \geqslant t_0 \geqslant \tau(\epsilon). \tag{2.8}$$

By applying Theorem 3.1.1 of [5], we get

$$V(t,x(t,t_0,x_0)) \leqslant r(t,t_0,u_0), \tag{2.9}$$

where $r(t,t_0,u_0)$ is the maximal solution of (1.6). Choose

$$V(t_0,x_0) = u_0. \tag{2.10}$$

From condition (viii), we get

$$\begin{aligned} b\|x(t,t_0,x_0)\| &\leqslant V(t,x_0) \leqslant r(t,t_0,u_0) \\ &= Mu_0 = MV(t_0,x_0) = ML\|x_0\| \\ &= N\|x_0\|, \quad N = ML, \end{aligned}$$

thus

$$\|x(t,t_0,x_0)\| \leqslant b^{-1}N\|x_0\|.$$

Hence

$$\|x(t,t_0,x_0)\| \leqslant \|x_0\|q(N) = Z\|x_0\|, \quad t \geqslant t_0 \geqslant \tau(\epsilon),$$

where $Z = q(N)$, $Z \leqslant 1$ is Lipschitzain constant. Thus, we have

$$\|x(t, t_0, x_0)\| \leqslant Z\|x_0\|, \quad t \geqslant t_0 \geqslant \tau(\epsilon),$$

provided that $\|x_0\| \leqslant \delta_1(\epsilon)$.

Hence the zero solution of (1.1) is uniformly eventually Lipschitz stable. □

Theorem 2.3. *If the zero solution of* (1.1) *is uniformly eventually Lipschitz stable such that*

$$\|\phi(t, s, x)h(s, x)\| \leqslant \gamma(s)\|x\|, \quad \int_{t_0}^{t} \gamma(s)\,\mathrm{d}s < \infty, \tag{2.11}$$

where $\phi(t, s, x)$ *is the fundamental matrix solution of* (1.11), *then the zero solution of* (1.2) *is uniformly eventually Lipschitz stable.*

Proof. Since the zero solution of (1.1) is uniformly eventually Lipschitz stable. Then for every $\epsilon > 0$, $t_0 \in J$, there exist $M > 1$, $\delta(\epsilon)$, and $\tau(\epsilon) > 0$ such that

$$\|x(t, t_0, x_0)\| \leqslant M\|x_0\|, \quad t \geqslant t_0 \geqslant \tau\epsilon, \tag{2.12}$$

where $\|x_0\| \leqslant \delta$. Let $y(t, t_0, x_0)$ and $x(t, t_0, x_0)$ be the solutions of (1.1) and (1.2) with the same initial values x_0, respectively. By using the variation of constant formula, the solutions of (1.1) and (1.2) with the same initial values are related by

$$x(t, t_0, x_0) = y(t, t_0, x_0) + \int_{t_0}^{t} \phi(t, s, x(s, t_0, x_0))h(s, x(s, t_0, x_0))\,\mathrm{d}s.$$

The rest of the proof is in the same line of the proof of Theorem 2.1. So, it is omitted. □

The following definition will be needed.

Definition 2.1. The zero solution of the system (1.1) is said to be *asymptotically stable in variation* if

$$\int_{t_0}^{t} \phi(s, t_0, x_0)\,\mathrm{d}s \leqslant M \tag{2.13}$$

for every $t_0 \geqslant 0$, and all $t \geqslant t_0$, where $\phi(t, t_0, x_0)$ is the fundamental matrix solution of (1.11) with $\phi(t_0, t_0, x_0) = E$.

Theorem 2.4. *Suppose the zero solution of* (1.1) *is uniformly eventually Lipschitz stable*, *and the zero solution of* (1.2) *is uniformly asymptotically stable in variation such that*

$$\|h(s, x(t, t_0, x_0))\| \leqslant \|x_0\|. \tag{2.14}$$

Then the zero solution of (1.2) *is uniformly eventually Lipschitz stable.*

Proof. As in Theorem 2.3, we have

$$\|x(t,t_0,x_0)\| = \|y(t,t_0,x_0)\| + \int_{t_0}^{t} \|\phi(t,s,x(s,t_0,x_0))h(s,x(s,t_0,x_0))\|\,\mathrm{d}s. \tag{2.15}$$

Since the zero solution of (1.2) is uniformly asymptotically stable in variation, we get

$$\int_{t_0}^{t} \phi(t,s,x(s,t_0,x_0))\,\mathrm{d}s \leqslant M \tag{2.16}$$

and since the zero solution of (1.1) is uniformly eventually Lipschitz stable, we have

$$\|y(t,t_0,x_0)\| \leqslant M\|x_0\|, \tag{2.17}$$

where $\|x_0\| \leqslant \delta$, $t \geqslant t_0 \geqslant \tau(\epsilon)$.

From (2.14)–(2.16), inequality (2.15) becomes

$$\begin{aligned}\|x(t,t_0,x_0)\| &= M\|x_0\| + \|x_0\| \int_{t_0}^{t} \|\phi(t,s,x(s,t_0,x_0))\|\,\mathrm{d}s \\ &\leqslant M\|x_0\| + M\|x_0\| = 2M\|x_0\| = N^*\|x_0\|,\end{aligned}$$

where $N^* = 2M$ is a Lipschitz constant. Then

$$\|x(t,t_0,x_0)\| \leqslant N^*\|x_0\|,$$

where $\|x_0\| \leqslant \delta$, $t \geqslant t_0 \geqslant \tau(\epsilon)$. Hence the zero solution of (1.2) is uniformly eventually Lipschitz stable. □

Theorem 2.5. *Let the zero solution of* (1.6) *be uniformly eventually Lipschitz stable, and* $g(t,u) \in C[J \times R^+, R^+]$, *and* $g(t,0) = 0$, *such that*

$$\|g(t,u) - g(t,v)\| \leqslant L\|u - v\|, \quad L > 1 \tag{2.18}$$

and

$$\|x + \delta f(t,x)\| \leqslant \|x\| + \delta g(t,\|x\|) + \epsilon(\delta) \tag{2.19}$$

for some positive constant L, and $(t,x) \in J \times \delta(\rho)$, *and for sufficiently small* $\delta > 0$, *with*

$$\mathrm{Lim}_{\delta\to 0}\frac{\epsilon(\delta)}{\delta} = 0.$$

Then the zero solution of (1.1) *is uniformly eventually Lipschitz stable.*

Proof. Let $V(t,x) = x$, and $x_0 = u_0$. Then it follows from (2.19) that

$$\begin{aligned} V' &= \mathrm{Lim}_{\delta\to 0}\frac{\|x+t+\delta\| - \|x(t)\|}{\delta} \\ &\leqslant \mathrm{Lim}_{\delta\to 0}\frac{1}{\delta}\|x+t+\delta\| + \delta g(t,\|x(t)\|) + \epsilon(\delta) + \|x+\delta f(t,x)\| \\ &\leqslant g(t,V). \end{aligned} \tag{2.20}$$

It follows from Theorem 3.1.1 of [5] that

$$V(t,x(t,t_0,x_0)) \leqslant r(t,t_0,u_0)$$

and thus

$$\|x(t,t_0,x_0)\| \leqslant V(t,x(t,t_0,x_0)) \leqslant r(t,t_0,u_0), \tag{2.21}$$

where $r(t,t_0,u_0)$ is the maximal solution of (1.6) through $(t_0,u_0) = (t_0,\|x_0\|)$. Since the zero solution of (1.6) is uniformly eventually Lipschitz stable, there exists $\alpha > 0$, $\delta(\epsilon) > 0$, and $\tau(\epsilon) > 0$ such that $u_0 = |u_0| \leqslant \delta$ implies

$$|u(t,t_0,u_0)| = u(t,t_0,\|x_0\|) \leqslant \alpha\|x_0\|.$$

By using (2.20), we obtain

$$\|x(t,t_0,x_0)\| \leqslant \alpha\|x_0\|, \quad t \geqslant t_0 \geqslant \tau,$$

whenever $\|x_0\| \leqslant \delta$, i.e., for $\epsilon > 0$, there exist $M > 0$, and $\delta > 0$ such that

$$\|x_0\| \leqslant \delta, \ x_0 \in R^n \quad \Rightarrow \quad \|x(t,t_0,x_0)\| \leqslant M\|x_0\|, \ t \geqslant t_0 > 0.$$

Then the zero solution of (1.1) is uniformly Lipschitz stable, and the proof is completed. □

3. Example

Now, we shall illustrate the results obtained by some examples:

Example 1. Consider the impulsive system

$$\begin{aligned} & y' = e^t y^2, \quad t \neq t_k, \\ & y(t_k^+) = c_k y(t_k), \quad 0 < c_k \leqslant 1, \end{aligned} \tag{3.1}$$

where c_k's are such that $\prod_{k=1}^{\infty} c_k = c_0 \in (0,1]$. Suppose that there exists a $\rho_0 > 0$ such that the moments of impulses $t-k$, and the impulses c_k's are related by

$$e^{-t_k} - e^{-t_{k-1}} \leqslant \frac{1-c_k}{k\rho_0}. \tag{3.2}$$

The solutions of (3.1) are given by

$$y(t,t_0^+,x_0) = \frac{\prod_{t_0<t_j<t} c_j x_0}{1 + x_0(e^{-t_1} - e^{-t_0}) + c_1(e^{-t_2} - e^{-t_1}) + \cdots + \prod_{t_0<t_j<t} c_j(e^{-t} - e^{-t_k})}$$

for $t \geqslant t_0$, $t \in (t_k, t_{k-1})$, and the fundamental matrix solution of the corresponding variational equation is

$$\phi(t,t_0^+,x_0) = \frac{\prod_{t_0<t_j<t} c_j}{\left[1 + x_0(e^{-t_1} - e^{-t_0}) + c_1(e^{-t_2} - e^{-t_1}) + \cdots + \prod_{t_0<t_j<tc_j}(e^{-t} - e^{-t_k})\right]^2}.$$

Choosing $V(s,x) = 2c_0x^2$, we get for $s \neq t_k$

$$D\_V(s, y(t,s,x)) = \frac{2c_0 y^2(t,s,x)}{\prod_{t_0<t_j<t} c_j} \leqslant [y(t,s,x)]^2 \leqslant [V(s,y(t,s,x))]^{3/2}$$

and

$$V(t_k^+, y(t,t_k^+,x(t_k^+))) = V(t_k^+, y(t,t_k^+,\beta_k x(t_k^+))) \leqslant 2c_0\beta_k^2 V(t_k, y(t,t_k,x(t_k))).$$

The corresponding comparison equation is

$$\begin{aligned} &u' = u^{\frac{3}{2}}, \quad t \neq t_k, \\ &u(t_k^+) = q_k^2 u(t_k), \quad q_k^2 = 2c_0\beta_k^2, \\ &u(t_0) = u_0 \geqslant 0. \end{aligned} \tag{3.3}$$

Assume that there exists $\rho_0 > 0$ such that the impulses q_k's and the moments of impulses t_k's are such that

$$t_{k-1} - t_k \leqslant \frac{2(1-q_k)}{qk\rho_0^{1/2}} \tag{3.4}$$

Let the zero solution of (3.1) be eventually stable. This follows from (3.2) and (3.4) which implies

$$\int_{t_k}^{t_{k-1}} \mathrm{e}^{-s}\,\mathrm{d}s + \int_{\sigma}^{q_k^2(\sigma)} \frac{\mathrm{d}s}{s(\frac{3}{2})} \leqslant 0$$

correspondingly, by using Lemma 1.1. Then from Theorem 2.1, we obtain the eventual stability of the zero solution of (3.1).

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